# Algebraic curves
# Solutions sheet 6

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Unless otherwise specified, $k$ is an algebraically closed field.

**Exercise 1.** Let $V$, $W$ be varieties and assume that $W$ is affine.

1. Show that there is a bijection $Hom_{Var}(V,W) \simeq Hom_{k-alg}(\mathcal{O}(W),\mathcal{O}(V))$.
2. Show that there is a bijection $\mathcal{O}(V) \simeq Hom_{Var}(V,\mathbb{A}^1_k)$.
3. Show that $\mathcal{O}(\mathbb{P}^1_k) \simeq k$.
4. Show that $\mathcal{O}(\mathbb{P}^n_k) \simeq k$ for all $n \geq 1$.

**Solution 1.**

1. Let $\phi : V \to W$ be a morphism of variety. Then $\phi^* \ : \ f \mapsto f \circ \phi$ define a $k-$algebra morphism $\mathcal{O}(W) \to \mathcal{O}(V)$.

   For the reverse bijection, as $W$ is affine, we can write $\mathcal{O}(W) = Spec\ A$, with $A$ a finite type $k-$algebra. Let $\psi \ : \ A \to \mathcal{O}(V)$ We can send it to

   $$F \ : \ x \in V \mapsto \mathfrak{m} := ker(f \mapsto \psi(f)(x))$$

   $f \mapsto \psi(f)(x)$ is a ring morphism $ev_x \ : \ A \to k$, so its kernel is a maximal ideal in $A$, hence a point of $W$. We have to check that this defines a morphism of varieties. It comes directly from the defintion, since for all $v \in V$, it implies that

   $$ev_{F(v)} = ev_v \cdot \psi$$

   and so we get a commutative diagram for all $f \in A$

   $$\begin{tikzcd} V \arrow[r, "F"] \arrow[dr, "\psi(f)"'] & W \arrow[d, "f"] \\ & k \end{tikzcd}$$

2. Clearly from 1. we get that $Hom_{Var}(V,\mathbb{A}^1) \simeq Hom_{k-alg}(k[x],\mathcal{O}(V))$. It suffices to check that the following is a reverse bijection :

   $$\begin{aligned} \mathcal{O}(V) &\longrightarrow Hom_{k-alg}(k[x],\mathcal{O}(V)) \\ f &\longmapsto (x \mapsto f) \\ \phi(x) &\longleftarrow \phi \end{aligned}$$

3. Using question 2, $\mathcal{O}(\mathbb{P}^1_k) = Hom_{Var}(\mathbb{P}^1, \mathbb{A}^1)$. Call $x_1$ and $x_2$ the projective corrdinates and $U_i$ the corresponding affine open. Let $f : \mathbb{P}^1 \to \mathbb{A}^1$. Then $f_{U_1}$ is polynomial in $\frac{x_2}{x_1}$ and $f_{U_2}$ is polynomial in $\frac{x_1}{x_2}$, so on the intersection, $f$ is constant. By continuity, $f \in k$.

4. Let $f : \mathbb{P}^n \to \mathbb{A}^1$. $f$ can be written uniquely as a quotient of coprime homogenous form of same degree, $f = \frac{P}{Q}$. Now, if $Q$ has positive degree, $Q$ has a zero $p$ (as we work over algebraically closed fields). Suppose $f$ can is defined at $p$. Then there are $P'$, $Q'$ such that $f' = \frac{P'}{Q'}$. Hence $PQ' = P'Q$, by coprimality, $P \mid P'$ so $P'$ also has a zero at $p$, which contradicts the assumption. So $f$ can not be defined. Hence $f$ is constant.

   We could also use that any two points in $\mathbb{P}^n$ lie on a $\mathbb{P}^1$.

**Exercise 2.** Let $n \geq 1$ and $f \in k[x_1, \ldots, x_n]$.

1. Show that $\mathbb{A}^n_k - V(f)$ is affine. What is its ring of regular functions?
2. Show that $\mathbb{A}^2_k - \{(0,0)\}$ is not affine. (Hint: compute the ring of regular functions).

**Solution 2.**

1. Let $D(f)$ denote $\mathbb{A}^n \setminus V(f)$. We have an obvious injection of rings :

$$\mathcal{O}(V(1 - Yf)) = k[x_1, \ldots, x_n][\frac{1}{f}] \longrightarrow \mathcal{O}(D(f)) \subset k(x_1, \ldots, x_n)$$

   Its inverse on points is :

$$\begin{aligned} V(1-Yf) \subset \mathbb{A}^{n+1} &\longrightarrow D(f) \subset \mathbb{A}^n \\ (\overline{x_1}, \ldots, \overline{x_n}, y) &\longmapsto (\overline{x_1}, \ldots, \overline{x_n}) \end{aligned}$$

   Indeed, $y \cdot f(\overline{x}) = 1$ implies that $f(\overline{x}) \neq 0$.

   Here $\overline{x}_i$ denotes the coordinates while $x_i$ denotes the variable in the ring of functions.

2. We can see that the ring of regular functions of $W := \mathbb{A}^2 - \{(0,0)\}$ is $k[x,y]$, the same as for $\mathbb{A}^2$, and so $W$ is not affine, because it is clearly not isomorphic to $\mathbb{A}^2$. Suppose that $f \in k[x,y]$ is invertible in $\mathcal{O}(W)$. As $k$ is algebraically closed, the zero locus of a polynomial of degree $d \geq 1$ is at least $1-$dimensional, so it can not be $\{(0,0)\}$. Thus $f \in k^*$ and $\mathcal{O}(W) = k[x,y]$.

**Exercise 3.** Let $\varphi : V \to W$ be a morphism of affine varieties and $\varphi^\sharp : \Gamma(W) \to \Gamma(V)$ the corresponding morphism of coordinate rings. Let $P \in V$ and $Q = \varphi(P)$ and consider local rings $\mathcal{O}_P(V)$, $\mathcal{O}_Q(W)$ with maximal ideals $\mathfrak{m}_P, \mathfrak{m}_Q$. Show that $\varphi^\sharp$ extends uniquely to a ring homomorphism $\mathcal{O}_Q(W) \to \mathcal{O}_P(V)$ and that $\varphi^\sharp(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$.

**Solution 3.** Recall that $\mathcal{O}_Q(W) \subset k(W)$ contains regular functions $\frac{f}{g}$ on $Q \in U \subset W$, such that $g(Q) \neq 0$. Hence $\phi^\sharp(g)(P) = g \circ \phi(P) \neq 0$. Thus the following unique extension of $\phi^\sharp$ is well-defined :

$$\begin{aligned} \mathcal{O}_Q(W) &\longrightarrow \mathcal{O}_P(V) \\ \tfrac{f}{g} &\longmapsto \tfrac{\phi^\sharp(f)}{\phi^\sharp(g)} \end{aligned}$$

$\frac{f}{g} \in \mathfrak{m}_Q$ implies $f(Q) = 0$ implies $\phi^\sharp(f)(P) = 0$ so that $\phi^\sharp(\frac{f}{g}) \in \mathfrak{m}_P$.

**Exercise 4.** Let $n \geq 1$ and $V$ a variety. We use projective coordinates $x_i,\ 1 \leq i \leq n+1$ on $\mathbb{P}^n_k$. Suppose there exist an open cover $(U_i)_{1 \leq i \leq n+1}$ of $V$ and morphisms of varieties $\varphi_i : U_i \to \{x_i \neq 0\} \subseteq \mathbb{P}^n_k,\ 1 \leq i \leq n+1$, such that $\forall i \neq j,\ (\varphi_i)_{\upharpoonright U_i \cap U_j} = (\varphi_j)_{\upharpoonright U_i \cap U_j}$. Show that there exists a unique morphism $\varphi : V \to \mathbb{P}^n_k$ such that $\varphi_{\upharpoonright U_i} = \varphi_i$. We say that $\varphi$ is obtained by *glueing* the $\varphi_i,\ 1 \leq i \leq n+1$.

**Solution 4.** We can clearly define

$$\phi(x) := \phi_i(x)$$

where $x \in V$ is such that $x \in U_i$. It is well-defined. We need to show that it is a morphism. Let $f \in \mathcal{O}_{\mathbb{P}^n}(W)$, a regular function on a proper open $W \subset \mathbb{P}^n$. Then define $W_i := W \cap \{x_i \neq 0\}$. On each $\phi^{-1}(W_i) \subseteq U_i$, we have $f \circ \phi = f \circ \phi_i$ is regular. A function is regular if it is regular at all points of $V$. For a fixed $P \in V$, $f \circ \phi$ is regular on at least one open $P \in W_i \subseteq W$. Hence $f \circ \phi$ is regular so $\phi$ is a morphism.

**Exercise 5.** Let $f \in k[x_1, x_2, x_3]$ an irreducible form of degree 2 and consider $V_P(f) \subseteq \mathbb{P}^2_k$.

1. Show that, up to a linear change of coordinates, we can assume that $f = x_2^2 - x_1 x_3$. (Hint: remember we classified similar subvarieties of $\mathbb{A}^2_k$).

2. Show that the map:

$$\begin{array}{ccc} \mathbb{P}^1_k & \to & \mathbb{P}^2_k \\ (s:t) & \mapsto & (s^2 : st : t^2) \end{array}$$

   induces an isomorphism $\mathbb{P}^1_k \simeq V_P(f)$. (Hint: take a look locally in the standard affine opens of projective space and use exercise 4).

**Solution 5.**

1. Consider $g(x,y) := f(1,x,y)\ :\ \mathbb{A}^2 \to k$. Then $V(g) \simeq V(f) \cap U_0$, with $U_0 = \{x_0 \neq 0\} \simeq \mathbb{A}^2$ as usual. Note that $V(f)$ is irreducible. By a previous exercise, we can use linear changes of coordinates to write

$$g(x,y) = x^2 - y$$

   or

$$g(x,y) = 1 - xy$$

   The projectivisation is then $x^2 - yz$ or $z^2 - xy$, and it is equal to $f$. We get the result by re-naming the variables if necessary.

2. We have isomorphisms on affine charts $U_0, U_1 \subset \mathbb{P}^1$, given by

$$U_0 \longrightarrow V_P(f) \cap U_0$$
$$\frac{x_1}{x_0} \mapsto \left(\frac{x_1}{x_0}, \left(\frac{x_1}{x_0}\right)^2\right)$$

   and

$$U_1 \longrightarrow V_P(f) \cap U_2$$
$$\frac{x_0}{x_1} \mapsto \left(\left(\frac{x_0}{x_1}\right)^2, \frac{x_0}{x_1}\right)$$

   They glue on $U_0 \cap U_2 \subset \mathbb{P}^2$.